ON A CONTACT PROBLEM FOR A VISCOELASTIC HALF-PLANE

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We solve the problem of pressure exerted by a die rigidly connected to an isotropic, viscoelastic half-plane, assuming that the viscoelastic material exhibits volume creep. We obtain an exact expression for the pressure existing under the die at any instant of time and thus establish not only the asymptotic value of the pressure, but also the kinetics of the process.

We shall write the stress-strain relations in an isotropic viscoelastic material in the following form [1] (the case of plane stress)

$$\varepsilon_x(t)=\frac{1}{E}\sigma_x(t)+\frac{1}{E}\int_0^t K_1(t-\tau)\sigma_x(\tau)\,d\tau-\frac{\nu}{E}\sigma_y(t)-\frac{\nu}{E}\int_0^t K_2(t-\tau)\sigma_y(\tau)\,d\tau$$

$$\varepsilon_y(t)=\frac{1}{E}\sigma_y(t)+\frac{1}{E}\int_0^t K_1(t-\tau)\sigma_y(\tau)\,d\tau-\frac{\nu}{E}\sigma_x(t)-\frac{\nu}{E}\int_0^t K_2(t-\tau)\sigma_x(\tau)\,d\tau$$

$$\gamma_{xy}(t)=\frac{1+\nu}{E}\tau_{xy}(t)+\frac{1+\nu}{E}\int_0^t K(t-\tau)\tau_{xy}(\tau)\,d\tau \qquad (1)$$

$$\left(K(t)=\frac{1}{1+\nu}K_1(t)+\frac{\nu}{1+\nu}K_2(t)\right)$$

We shall consider the problem of pressure exerted by a plane, rectangular, rigid die, on an isotropic viscoelastic material. Let the die be rigidly connected to a half-plane. The die is acted upon by external forces whose resultant coincides with the y-axis, so that $X=0$ and $Y=-P_0$ where P_0 is a given positive constant. We assume that the surface of the viscoelastic material outside the die is stress-free.

We shall utilize a solution of this problem for an elastic half-plane given in [2]. For the case of an isotropic viscoelastic material possessing the property of volume creep, expressions for the pressure $P(p,x)$ and the tangential stress $T(p,x)$ are

$$P(p,x)=\frac{P_0}{\pi\sqrt{l^2-x^2}}\frac{1+\varkappa(p)}{\sqrt{\varkappa(p)}}\cos\left[\frac{\ln\varkappa(p)}{2\pi}\ln\frac{l+x}{l-x}\right] \qquad (2)$$

$$T(p,x)=\frac{P_0}{\pi\sqrt{l^2-x^2}}\frac{1+\varkappa(p)}{\sqrt{\varkappa(p)}}\sin\left[\frac{\ln\varkappa(p)}{2\pi}\ln\frac{l+x}{l-x}\right]$$

Here $\varkappa$ ($\varkappa$ is a constant introduced by Muskhelishvili) is given by

$$\varkappa(p)=\frac{3-\nu^*(p)}{1+\nu^*(p)}, \qquad \nu^*(p)=\nu\frac{1+K_2(p)/p}{1+K_1(p)/p} \qquad (3)$$

Making the notation

$$\alpha=\frac{1}{2\pi}\ln\frac{l+x}{l-x}$$

we can transform the expression for pressure given in (2), into

$$P(p,x)=\frac{P_0}{2\pi\sqrt{l^2-x^2}}\left[\varkappa(p)^{-1/2+i\alpha}+\varkappa(p)^{1/2-i\alpha}+\varkappa(p)^{-1/2-i\alpha}+\varkappa(p)^{1/2-i\alpha}\right] \qquad (4)$$

Thus we reduce the problem to that of finding the initial function for the transform

$[\varkappa(p)]^\gamma$ where γ is the power index accompanying $\varkappa(p)$ in (4).

Experimental data available for numerous materials indicate that creep kernels can be approximated sufficiently well using an exponential function. We shall therefore assume that in (1)

$$K_i(t-\tau) = k_i e^{-\beta(t-\tau)} \qquad (i=1,2)$$

Using (3) we find

$$\varkappa(p) = \frac{ap+b}{cp+d} \qquad \begin{pmatrix} a=3-\nu, & b=3(\beta+k_1)-\nu(\beta+k_2) \\ c=1+\nu, & d=\beta+k_1+\nu(\beta+k_2) \end{pmatrix}$$

The problem is thus reduced to obtaining the inverse transform for

$$\left[\frac{ap+b}{cp+d}\right]^\gamma = \left(\frac{a}{c}\right)^\gamma \left(\frac{p+\lambda}{p+\lambda-\alpha}\right)^\gamma \qquad \left(\lambda=\frac{b}{a},\ \alpha=\frac{b}{a}-\frac{d}{c}\right)$$

The inverse transform for $[p/(p-\alpha)]^\gamma$ can be written in the form of a degenerate hypergeometric function

$$\sum_{k=0}^{\infty} \frac{(\gamma+k-1)\ldots\gamma}{k!}\alpha^k \frac{t^k}{k!}$$

convergent for any t. This enables us to obtain the inverse transform for $[\varkappa(p)]^\gamma = [(ap+b/(cp+d)]^\gamma$ in the following form:

$$\left(\frac{a}{c}\right)^\gamma \frac{b}{a} \sum_{k=0}^{\infty} \frac{(\gamma+k-1)\ldots\gamma}{(k!)^2}\left(\frac{b}{a}-\frac{d}{c}\right)^k \left[\left(\frac{a}{b}\right)^{k+1} k! - \right.$$
$$\left. - e^{-bt/a} \sum_{n=1}^{k} \left(\frac{a}{b}\right)^{n+1} k(k-1)\ldots(k-n+1)\,t^{k-n}\right]$$

We can thus obtain the expression for the pressure $P(t,x)$ arising under the die acting upon a viscoelastic half-plane in the form of a sum of four series, each of them convergent for any t. Using the gamma function relationships we finally obtain

$$P(t,x) = \frac{P_0}{\pi\sqrt{l^2-x^2}} \frac{b}{a} \sum_{k=0}^{\infty} \left\{ \mathrm{Re}\left[\frac{\Gamma(-{}^1/_2+i\alpha+k)}{\Gamma(-{}^1/_2+i\alpha)}\left(\frac{a}{c}\right)^{-{}^1/_2+i\alpha} \times \right.\right. \tag{5}$$
$$\times \left.\left(\frac{a}{c}\frac{-{}^1/_2+i\alpha+k}{-{}^1/_2+i\alpha}+1\right)\right] \frac{(b/a-d/c)k}{(k!)^2}\left[\left(\frac{a}{b}\right)^{k+1} k! - \right.$$
$$\left.\left. - e^{-bt/a}\sum_{n=1}^{k}\left(\frac{a}{b}\right)^{n+1} k\ldots(k-n+1)\,t^{k-n}\right]\right\}$$
$$\left(\alpha = \frac{1}{2\pi}\ln\frac{l+x}{l-x}\right)$$

We can investigate this solution for various durations of applied load and for various positions under the die. Near the corners of the die, $-l$ and $+l$, the stresses change their sign an infinite number of times just as in the case of the elastic problem, the size of these segments is however very small.

If the loading is instantaneous, i.e. if $t \to 0$ in (5), the result found coincides with the solution obtained for an elastic half-plane [2, 3].

For the extended time intervals we have the following asymptotic expression:

$$P(x) = \frac{P_0}{\pi\sqrt{l^2-x^2}} \mathrm{Re}\left\{\left(\frac{a}{c}\right)^{-{}^1/_2+i\alpha} \sum_{k=0}^{\infty}\left(-\frac{1}{2}+i\alpha+k-1\right)\ldots\left(-\frac{1}{2}+i\alpha\right)\times\right.$$
$$\left.\times\left(\frac{a}{c}\frac{-{}^1/_2+i\alpha+k}{-{}^1/_2+i\alpha}+1\right)\frac{(1-ad/bc)^k}{k!}\right\}$$

This can be written as a sum of three hypergeometric functions

$$P(x)=\frac{P_0}{\pi\sqrt{l^2-x^2}}\operatorname{Re}\left[\left(\frac{a}{c}\right)^{1/2+i\alpha}F\left(-\frac{1}{2}+i\alpha,1;1;1-\frac{ad}{bc}\right)+\right.\tag{6}$$
$$+\left(\frac{a}{c}\right)^{1/2+i\alpha}\left(1-\frac{ad}{bc}\right)F\left(\frac{1}{2}+i\alpha,1;1;1-\frac{ad}{bc}\right)+$$
$$\left.+\left(\frac{a}{c}\right)^{-1/2+i\alpha}F\left(-\frac{1}{2}+i\alpha,1;1;1-\frac{ad}{bc}\right)\right]$$

Pressure under the center of the die ($x=0$) is given by

$$P(t,0)=\frac{P_0}{\pi l}\frac{b}{a}\left(\frac{a}{c}\right)^{-1/2}\sum_{k=0}^{\infty}\left(-\frac{1}{2}+k-1\right)\ldots\left(-\frac{1}{2}\right)\left(\frac{a}{c}-\frac{2ka}{c}+1\right)\times$$
$$\times\frac{(b/a-d/c)^k}{(k!)^2}\left[\left(\frac{a}{b}\right)^{k+1}k!-e^{-bt/a}\sum_{n=1}^{k}\left(\frac{a}{b}\right)^{n+1}k\ldots(k-n+1)\,t^{k-n}\right]$$

while in the case of an instantaneous loading we have

$$P(0,0)=\frac{P_0}{\pi l}\frac{\varkappa+1}{\sqrt{\varkappa}}\tag{7}$$

When the load is applied for extended periods, the pressure varies monotonically and tends to a limit

$$\underset{t\to\infty}{P}(t,0)=\frac{P_0}{\pi l}\left\{\left[\left(\frac{a}{c}\right)^{1/2}+\left(\frac{a}{c}\right)^{-1/2}\right]F\left(-\frac{1}{2},1;1;1-\frac{ad}{bc}\right)+\right.$$
$$\left.+\left(\frac{a}{c}\right)^{1/2}\left(1-\frac{ad}{bc}\right)F\left(\frac{1}{2},1;1;1-\frac{ad}{bc}\right)\right\}$$

Using the expressions for certain hypergeometric functions we can simplify the above expression to

$$\underset{t\to\infty}{P}(t,0)=\frac{P_0}{\pi l}\frac{b/d+1}{\sqrt{b/d}}\tag{8}$$

where the ratio b/d is given in terms of the material constants as follows:

$$\frac{b}{d}=\frac{3(\beta+k_1)-\nu(\beta+k_2)}{\beta+k_1+\nu(\beta+k_2)}=\frac{3-\nu\eta}{1+\nu\eta}\qquad\left(\eta=\frac{\beta+k_2}{\beta+k_1}\right)$$

As expected, the expression (8) which gives the pressure when $t\to\infty$ resembles (7) which is valid for the initial instant of time, but the value of $\varkappa$ is different in each case. Comparing (7) and (8) we find, that the pressure under the center of the die increases with time when $\eta<1$ and $\eta>2\nu^{-1}-1$ and decreases when $1<\eta<2\nu^{-1}-1$. (Poisson's ratio varies for real materials within the range $0<\nu<1/2$).

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